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LETTER TO THE EDITOR

Universal R-matrix for non-standard quantum $sl(2, \mathbb{R})$

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Abstract. A universal R-matrix for the non-standard (Jordanian) quantum deformation of $sl(2, \mathbb{R})$ is presented. A family of solutions of the quantum Yang–Baxter equation is obtained from some finite-dimensional representations of this Lie bialgebra quantization of $sl(2, \mathbb{R})$.

The quantum Yang–Baxter equation (YBE)

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}. \tag{1}$$

was discovered to play a relevant role as the integrability condition for $(1+1)$ quantum field theories [1] and also in connection with two-dimensional models in lattice statistical physics [2], conformal field theory [3] and knot theory [4]. It is now well known that investigation of the algebraic properties of this equation and the obtaining of new solutions are closely related to the study of quantum groups and algebras [5].

In fact, let $\mathcal{A}$ be a Hopf algebra and let $\mathcal{R}$ be an invertible element in $\mathcal{A} \otimes \mathcal{A}$ such that

$$\sigma \circ \Delta(X) = \mathcal{R}\Delta(X)\mathcal{R}^{-1} \qquad \forall\, X \in \mathcal{A} \tag{2}$$

where σ is the flip operator $\sigma(x \otimes y) = (y \otimes x)$. If we write $\mathcal{R} = \sum_i a_i \otimes b_i$, $\mathcal{R}_{12} \equiv \sum_i a_i \otimes b_i \otimes 1$, $\mathcal{R}_{13} \equiv \sum_i a_i \otimes 1 \otimes b_i$, $\mathcal{R}_{23} \equiv \sum_i 1 \otimes a_i \otimes b_i$ and the relations

$$(\Delta \otimes id)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23} \qquad (id \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12} \tag{3}$$

are fulfilled, $(\mathcal{A}, \mathcal{R})$ is called a *quasitriangular* Hopf algebra [6]. In that case, $\mathcal{R}$ is easily proven to be a solution of (1). Hereafter, if $\mathcal{A}$ is a Hopf algebra and $\mathcal{R}$ fulfills both (1) and (2), we shall say that $\mathcal{R}$ is a (quantum) *universal R-matrix* for $\mathcal{A}$. Obviously, different representations for the algebra $\mathcal{A}$ will give rise to different explicit solutions of the quantum YBE.

In particular, let $\mathcal{A}$ be a quantum deformation $U_z(g)$ of the universal enveloping algebra of a Lie algebra g. Then, the quasicocommutativity property (2) is translated, in terms of the Hopf algebra dual to $\mathcal{A}$, into the known FRT relations defining the quantum group $Fun_z(G)$ [5]. It is also known that each $U_z(g)$ defines a unique Lie bialgebra structure on g that can be used to characterize the quantum deformation. For all semisimple Lie algebras, all these Lie bialgebra structures are coboundaries generated by classical r-matrices. In $sl(2, \mathbb{R})$, two outstanding Lie bialgebra structures can be mentioned: the standard one, generated by the classical r-matrix $r^{(s)} = \lambda\, J_+ \wedge J_-$ and the non-standard (triangular) Lie bialgebra given by the element $r^{(n)} = \chi\, J_3 \wedge J_+$ $(\lambda, \chi \in \mathbb{R})$. The quantization of the former is the well known Drinfel'd–Jimbo deformation, whose quantum universal R-matrix was given in [7]. For the latter, the corresponding non-standard quantum algebra is the so-called Jordanian deformation of $sl(2,\mathbb{R})$ [8] (introduced for the first time in [9] in a quantum group setting; see [10] and references therein and [11], where the non-standard deformation of $so(2,2)$ was constructed). To our knowledge, no quantum universal R-matrix is known for this case (the R given in [8] is neither a solution of (1) nor verifies (2), as it has already been pointed out in [12, 13]). The aim of this letter is to provide it.

Let us consider the $sl(2,\mathbb{R})$ Lie algebra with the following commutation relations

$$[J_3, J_+] = 2J_+ \qquad [J_3, J_-] = -2J_- \qquad [J_+, J_-] = J_3. \tag{4}$$

The classical r-matrix

$$r = z\, J_3 \wedge J_+ \tag{5}$$

is a solution of the classical YBE and it generates the Lie bialgebra structure with cocommutators given, as usual, by $\delta(X) = [1\otimes X + X\otimes 1, r]$:

$$\delta(J_+) = 0 \qquad \delta(J_3) = 2\,z\, J_3 \wedge J_+ \qquad \delta(J_-) = 2\,z\, J_- \wedge J_+. \tag{6}$$

This map is the first order (in the deformation parameter z) of the co-antisymmetric part of the deformed coproduct corresponding to the non-standard quantization of $sl(2,\mathbb{R})$ given [8] by the relations

$$\begin{aligned}
\Delta(J_+) &= 1\otimes J_+ + J_+\otimes 1\\
\Delta(J_3) &= \mathrm{e}^{-zJ_+}\otimes J_3 + J_3\otimes \mathrm{e}^{zJ_+}\\
\Delta(J_-) &= \mathrm{e}^{-zJ_+}\otimes J_- + J_-\otimes \mathrm{e}^{zJ_+}
\end{aligned} \tag{7}$$

$$[J_3, J_+] = 2\,\frac{\sinh(zJ_+)}{z} \qquad [J_+, J_-] = J_3 \tag{8}$$

$$[J_3, J_-] = -J_-\cosh(zJ_+) - \cosh(zJ_+)J_-. \tag{9}$$

In order to obtain a universal R-matrix linked to this quantum algebra, an essential point is to choose an adequate basis for the deformation. Let us perform a (nonlinear) transformation of the generators $\{J_3, J_+, J_-\}$ as follows:

$$\begin{aligned}
&A_+ = J_+ \qquad A = \mathrm{e}^{zJ_+}J_3\\
&A_- = \mathrm{e}^{zJ_+}J_- - \tfrac{1}{4}z\mathrm{e}^{zJ_+}\sinh(zJ_+).
\end{aligned} \tag{10}$$

This change of basis leads to a Hopf algebra (denoted here by $U_z sl(2,\mathbb{R})$) with the following coproduct (Δ), counit (ϵ), antipode (γ) and commutation rules:

$$\begin{aligned}
\Delta(A_+) &= 1\otimes A_+ + A_+\otimes 1\\
\Delta(A) &= 1\otimes A + A\otimes \mathrm{e}^{2zA_+}\\
\Delta(A_-) &= 1\otimes A_- + A_-\otimes \mathrm{e}^{2zA_+}
\end{aligned} \tag{11}$$

$$\epsilon(X) = 0 \qquad \text{for } X\in\{A, A_+, A_-\} \tag{12}$$

$$\gamma(A_+) = -A_+ \qquad \gamma(A) = -A\mathrm{e}^{-2zA_+} \qquad \gamma(A_-) = -A_-\mathrm{e}^{-2zA_+} \tag{13}$$

$$[A, A_+] = \frac{\mathrm{e}^{2zA_+}-1}{z} \qquad [A, A_-] = -2A_- + zA^2 \qquad [A_+, A_-] = A. \tag{14}$$

The quantum Casimir belonging to the centre of $U_z sl(2,\mathbb{R})$ is now

$$\mathcal{C}_z = \frac{1}{2}A\,\mathrm{e}^{-2zA_+}A + \frac{1-\mathrm{e}^{-2zA_+}}{2z}A_- + A_-\frac{1-\mathrm{e}^{-2zA_+}}{2z} + \mathrm{e}^{-2zA_+} - 1. \tag{15}$$

Obviously, both Lie bialgebras generated by $\{J_3, J_+, J_-\}$ and by $\{A, A_+, A_-\}$ are formally identical, and the classical r-matrix associated to $U_z sl(2,\mathbb{R})$ is also $r = z\, A \wedge A_+$.

Now it is important to realize that $\{A, A_+\}$ generates a Hopf subalgebra. Moreover, this Hopf subalgebra has a known universal R-matrix, as proven in [14].

The main result of this letter can be stated as follows.

Proposition. The element

$$\mathcal{R} = \exp\{-zA_+ \otimes A\} \exp\{zA \otimes A_+\}. \tag{16}$$

is a quantum universal R-matrix for $U_z sl(2,\mathbb{R})$.

Proof. The element (16) coincides with the universal R-matrix for the Hopf subalgebra $\{A, A_+\}$ given in [14]. That (16) is a solution of the quantum YBE is a consequence of the properties of the universal T-matrix [15] from which $\mathcal{R}$ was obtained. Thus, we only need to prove that property (2) is fulfilled for the remaining generator A_-. In order to compute the right-hand side of (2) we have to take into account that

$$\mathrm{e}^f\, \Delta(A_-)\, \mathrm{e}^{-f} = \Delta(A_-) + \sum_{n=1}^{\infty} \frac{1}{n!}\, [f, \ldots [f, \Delta(A_-)]^{n)} \ldots]. \tag{17}$$

By setting $f = z\, A \otimes A_+$ we find

$$\begin{aligned}
&[zA \otimes A_+, \Delta(A_-)] = zA \otimes A + z(-2A_- + zA^2) \otimes A_+ \mathrm{e}^{2zA_+} \\
&[zA \otimes A_+, [zA \otimes A_+, \Delta(A_-)]] = zA^2 \otimes (1 - \mathrm{e}^{2zA_+}) - 2z^2(-2A_- + zA^2) \otimes A_+^2 \mathrm{e}^{2zA_+}.
\end{aligned} \tag{18}$$

It is now easy to check that, for $n \geqslant 3$,

$$[zA \otimes A_+, \ldots [zA \otimes A_+, \Delta(A_-)]^{n)} \ldots] = (-2)^{n-1} z^n (-2A_- + zA^2) \otimes A_+^n e^{2zA_+}. \tag{19}$$

Therefore, if we call the following expression g:

$$\begin{aligned}
\exp\{zA \otimes A_+\}\Delta(A_-) \exp\{-zA \otimes A_+\} &= \Delta(A_-) + zA \otimes A + \tfrac{1}{2} zA^2 \otimes (1 - \mathrm{e}^{2zA_+}) \\
&\quad - \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-2z)^n}{n!} (-2A_- + zA^2) \otimes A_+^n e^{2zA_+} \\
&= 1 \otimes A_- + A_- \otimes \mathrm{e}^{2zA_+} + zA \otimes A + \tfrac{1}{2} zA^2 \otimes (1 - \mathrm{e}^{2zA_+}) \\
&\quad - \tfrac{1}{2}(-2A_- + zA^2) \otimes (\mathrm{e}^{-2zA_+} - 1)\mathrm{e}^{2zA_+} \\
&= 1 \otimes A_- + A_- \otimes 1 + zA \otimes A = g
\end{aligned} \tag{20}$$

and we compute (17) with $f = -z\, A_+ \otimes A$ and g instead of $\Delta(A_-)$, we shall obtain

$$\begin{aligned}
&[-zA_+ \otimes A, g] = -zA_+ \otimes (-2A_- + zA^2) - zA \otimes A - z(1 - \mathrm{e}^{2zA_+}) \otimes A^2 \\
&[-zA_+ \otimes A, [-zA_+ \otimes A, g]] = -2z^2 A_+^2 \otimes (-2A_- + zA^2) + z(1 - \mathrm{e}^{2zA_+}) \otimes A^2
\end{aligned} \tag{21}$$

and, for $n \geqslant 3$,

$$[-zA_+ \otimes A, \ldots [-zA_+ \otimes A, g]^{n)} \ldots] = -2^{n-1} z^n A_+^n \otimes (-2A_- + zA^2). \tag{22}$$

The proof now follows:

$$
\begin{aligned}
\exp\{-zA_+ \otimes A\}\, g\, \exp\{zA_+ \otimes A\} = g - zA \otimes A - \tfrac{1}{2}z(1 - \mathrm{e}^{2zA_+}) \otimes A^2 \\
-\frac{1}{2}\sum_{n=1}^{\infty} \frac{(2zA_+)^n}{n!} \otimes (-2A_- + zA^2) \\
&= 1 \otimes A_- + A_- \otimes 1 - \tfrac{1}{2}z(1 - \mathrm{e}^{2zA_+}) \otimes A^2 - \tfrac{1}{2}(\mathrm{e}^{2zA_+} - 1) \otimes (-2A_- + zA^2) \\
&= A_- \otimes 1 + \mathrm{e}^{2zA_+} \otimes A_- \\
&= \sigma \circ \Delta(A_-).
\end{aligned} \tag{23}
$$

Note that $\mathcal{R}^{-1} = \mathcal{R}_{21}$ and, therefore, $\mathcal{R}$ is a triangular R-matrix.

A short digression concerning some representations of $U_z sl(2, \mathbb{R})$ can now be meaningful. Firstly, the two-dimensional matrix representation of $sl(2, \mathbb{R})$ defined by

$$
D(A) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad D(A_+) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad D(A_-) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{24}
$$

is also a matrix representation for $U_z sl(2, \mathbb{R})$ and it can be used in the FRT approach [5] to obtain the quantum group $Fun_z(SL(2, \mathbb{R})$. In such a representation, the universal R-matrix (16) has the form

$$
\begin{aligned}
D(\mathcal{R}) &= I \otimes I + zD(A) \otimes D(A_+) - zD(A_+) \otimes D(A) - z^2 D(A_+)D(A) \otimes D(A)D(A_+) \\
&= \begin{pmatrix} 1 & z & -z & z^2 \\ 0 & 1 & 0 & z \\ 0 & 0 & 1 & -z \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\end{aligned} \tag{25}
$$

This is precisely the R-matrix introduced by Zakrzewski in [16] and the corresponding quantum $SL(2, \mathbb{R})$ group is that given in [16, 8].

In contrast, the three-dimensional matrix representation of $U_z sl(2, \mathbb{R})$ does not coincide with the classical one and, in this sense, it can be considered as the first 'non-trivial' case. By assuming that the deformed matrix realization of the primitive generator is

$$
D_z(A_+) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{26}
$$

and by imposing the commutation rules (14) to be fulfilled, a straightforward computation leads to the matrices

$$
D_z(A) = \begin{pmatrix} 2 & a & b \\ 0 & 0 & a - 2z \\ 0 & 0 & -2 \end{pmatrix} \qquad D_z(A_-) = \begin{pmatrix} -a + 2z & c & d \\ 2 & 2z & b + c \\ 0 & 2 & a \end{pmatrix} \tag{27}
$$

where a and b are arbitrary and c, d are given by

$$
\begin{aligned}
c &= \tfrac{1}{4}(2az - 2b - a^2) \\
d &= \tfrac{1}{4}(a^2 z + 2bz - 2az^2 - 2ab).
\end{aligned} \tag{28}
$$

Since $D_z(A_+)^3$ vanishes, the universal R-matrix (16) is realized as

$$D_z(\mathcal{R}) = (1 - z D_z(A_+) \otimes D_z(A) + \tfrac{1}{2} z^2 D_z(A_+)^2 \otimes D_z(A)^2) \times (1 + z D_z(A) \otimes D_z(A_+) + \tfrac{1}{2} z^2 D_z(A)^2 \otimes D_z(A_+)^2). \tag{29}$$

Explicitly, $D_z(\mathcal{R})$ is the following solution of the quantum YBE

$$= \begin{pmatrix} 1 & 2z & 2z^2 & -2z & 0 & -bz + az^2 & 2z^2 & bz - az^2 & 0 \\ 0 & 1 & 2z & 0 & 0 & 2z^2 & 0 & 0 & bz - az^2 + 2z^3 \\ 0 & 0 & 1 & 0 & 0 & 2z & 0 & 0 & 2z^2 \\ 0 & 0 & 0 & 1 & 0 & 0 & -2z & 2z^2 & -bz + az^2 - 2z^3 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2z \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2z & 2z^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2z \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{30}$$

Actually, if we define $p = -bz + az^2$, we can reduce this solution to a two-parameter R-matrix.

Note also that a differential realization of the commutation rules (14) (with $\lambda(\lambda + \frac{1}{2})$ as the induced eigenvalue of the quantum Casimir (15)) can be obtained by defining

$$\begin{aligned} A &= \frac{e^{2zx} - 1}{z} \partial_x - \lambda \frac{e^{2zx} + 1}{2} \qquad A_+ = x \\ A_- &= -\frac{e^{2zx} - 1}{2z} \partial_x^2 + \lambda \frac{e^{2zx} + 1}{2} \partial_x - z \lambda^2 \frac{e^{2zx} - 1}{8}. \end{aligned} \tag{31}$$

The $z \to 0$ limit of (31) is the usual second-order differential realization of $sl(2, \mathbb{R})$

$$A = 2x\,\partial_x - \lambda \qquad A_+ = x \qquad A_- = -x\,\partial_x^2 + \lambda\,\partial_x \tag{32}$$

with the same eigenvalue $\lambda(\lambda + \frac{1}{2})$ coming from the classical limit of (15).

Finally, we would like to stress again the fact that the use of different basis of the same quantum algebra can be helpful in order to find the associated quantum R-matrix. On the other hand, a complete and systematic study of the representation theory of this non-standard deformation seems worth to be done in order to find new matrix solutions of the quantum YBE linked to $U_z sl(2, \mathbb{R})$.

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Note added in proof. After submission of this letter we were informed of the paper [17], where another expression for a quantum R-matrix of the non-standard deformation of $sl(2, \mathbb{R})$ has been given.

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